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A GENERALIZATION OF THE INGLETON—MAIN LEMMA AND A CLASS OF NON-ALGEBRAIC MATROIDS

B. LINDSTRÖM

Dedicated to professor Jan-Erik Roos on his fifthieth birthday

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Let K be an algebraically closed field of finite transcendence degree over an algebraically closed subfield F. The algebraically closed fields between F and K ordered by containment give a geometric lattice. The rank of a flat in this lattice is the transcendence degree over F of the corresponding field. On the atoms of the lattice we get a combinatorial geometry as in [1], which is called an ACCG (algebraically closed combinatorial geometry).

When Ingleton and Main found the first example of a non-algebraic matroid (1975) their proof depended on a lemma for ACCGs, which we want to generalize in order to find new examples of non-algebraic matroids. We shall prove the following generalization of the Ingleton—Main lemma.

Let π_1, π_2, π_3 be three flats of an ACCG over a field F such that for an integer $r \geqq 3$: $r(\pi_1 \vee \pi_2 \vee \pi_3) = r$, $r(\pi_i \vee \pi_j) = r-1$ $(1 \leqq i < j \leqq 3)$ and $r(\pi_i) = r-2$ $(1 \leqq i \leqq 3)$. Then we have $\pi_1 \wedge \pi_2 = \pi_1 \wedge \pi_3 = \pi_2 \wedge \pi_3$ and the rank of this flat is $r-3$.

As an application we obtain an infinite class of non-algebraic matroids such that no member of this class is a minor of another member.

We assume some familiarity with combinatorial geometries [1] and matroid theory [6]. It is well-known that algebraic independence satisfies the axioms of matroid theory [6, Chapter 11].

Consider an algebraically closed field K of finite transcendence degree over an algebraically closed subfield F. The algebraically closed fields between F and K then give a geometric lattice [1, Proposition 3.3]. The rank of a flat in this lattice is the transcendence degree of the corresponding field over F. Flats of rank 1 are called points, flats of rank 2 lines and flats of rank 3 are called planes. The combinatorial geometry on the atoms is called an ACCG for brevity.

Example. Let $F(x, y, z)$ be the field of all rational functions of three algebraically independent transcendentals x, y, z over F. The algebraic closure of this field is denoted by $\overline{F(x, y, z)}$ or briefly $\overline{x, y, z}$. The subfield $\overline{F(x)} = \bar{x}$ is a point. The subfield $\overline{F(x, y)} = \overline{x, y}$ is a line. The point $\bar{x}$ belongs to the line $\overline{x, y}$. Note that two lines in a plane do not always meet in a point. The lines $\overline{x, y}$ and $\overline{xz+y, z}$ in the plane $\overline{x, y, z}$ is an example, which was discussed in [3]. The following lemma gives a sufficient condition for lines to meet in a point.

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Ingleton—Main lemma

Let l_1, l_2, l_3 be three lines of an ACCG of rank at least 4. Assume that any two of these lines are coplanar, but not all three. Then the lines will meet in a point. We shall prove the following generalization.

Theorem. *Let* π_1, π_2, π_3 *be three flats of an ACCG over* F *such that* $r(\pi_1 \vee \pi_2 \vee \pi_3) = r \geqq 3$, $r(\pi_i \vee \pi_j) = r-1$ $(1 \leqq i < j \leqq 3)$ *and* $r(\pi_i) = r-2$ $(1 \leqq i \leqq 3)$. *Then* $\pi_1 \wedge \pi_2 = \pi_1 \wedge \pi_3 = \pi_2 \wedge \pi_3$ *and the rank of this flat is* $r-3$.

The proof will be elementary using an idea of Lovász [4], who suggested a proof of the Ingleton—Main lemma using resultants.

Lemma 1. *Let* $f_i \in k[X, Y]$ *for* $1 \leqq i \leqq n$ *have* $GCD = 1$. *Then these* $f_i(X, Y)$ *have at most a finite number of common zeroes in any extension of the field* k. ∎

Lemma 2. *Let* $f, g \in k[X]$ *have the resultant* $R(f, g)$. *Then* $GCD(f, g) \neq 1$ *if and only if* $R(f, g) = 0$. ∎

Proofs of Lemma 1 and Lemma 2 can be found e.g. in [5].

Lemma 3. *Let* π_1, π_2, π_3 *be elements of a lattice such that* $(\pi_i \vee \pi_j) \wedge (\pi_i \vee \pi_k) = \pi_i$ *when* $\{i, j, k\} = \{1, 2, 3\}$. *Then it follows* $\pi_1 \wedge \pi_2 = \pi_1 \wedge \pi_3 = \pi_2 \wedge \pi_3$.

Proof. $\pi_j \wedge \pi_k \leqq (\pi_i \vee \pi_j) \wedge (\pi_i \vee \pi_k) = \pi_i$ implies $\pi_j \wedge \pi_k \leqq \pi_1 \wedge \pi_2 \wedge \pi_3$ for $1 \leqq j < k \leqq 3$. Hence $\pi_j \wedge \pi_k = \pi_1 \wedge \pi_2 \wedge \pi_3$. ∎

Proof of the theorem. The proof is by induction over r when $r \geqq 3$. The case $r=3$ is trivial. Then assume that $r>3$.

Observe that the conditions of Lemma 3 are satisfied. Therefore we have $\pi_1 \wedge \pi_2 = \pi_1 \wedge \pi_3 = \pi_2 \wedge \pi_3 (= \pi_4)$. We prove first that π_4 contains a point. Choose $\bar{a} \neq \bar{b}$ in π_1. If $\bar{a}$ or $\bar{b}$ belongs to $\pi_2 \cup \pi_3$. Then we have our point. Therefore we may assume that $\bar{a}, \bar{b} \notin \pi_2 \cup \pi_3$. Since $r(\pi_1 \vee \pi_2 \vee \pi_3) - r(\pi_2 \vee \pi_3) = 1$ we may choose $\bar{b} \notin \pi_2 \vee \pi_3$. Since $r(\pi_1 \vee \pi_2) - r(\pi_2) = 1$ it follows that a and b are algebraically dependent over the field π_2. Therefore there is a polynomial $P(X, Y)$ with coefficients in the field π_2 such that $P(a, b) = 0$. Similarly, there is a polynomial $Q(X, Y)$ with coefficients in the field π_3 such that $Q(a, b)$. Note that $P(X, Y)$ and $Q(X, Y)$ contain both variables explicitely since $\bar{a}, \bar{b} \notin \pi_2 \cup \pi_3$. Let $R(Y)$ be the resultant which eliminates X between $P(X, Y)$ and $Q(X, Y)$.

Since $P(X, b)$ and $Q(X, b)$ have a common zero, $X=a$, it follows, by Lemma 2, $R(b)=0$. By our choice b does not depend on $\pi_2 \vee \pi_3$. $R(Y)$ has coefficients in $\pi_2 \vee \pi_3$. Therefore $R(Y)=0$ identically. It follows that $P(X, c)$ and $Q(X, c)$ have a common zero $\alpha(c)$ for any $c \in F$. We want to find $c \in F$ such that $\alpha(c)$ is transcendental over F. In order to see that this is possible, we consider $P(X, Y)$ in some more detail.

Since a, b are algebraically dependent over the field π_2, there is a circuit $\{\bar{a}, \bar{b}, \bar{c}_1, \ldots, \bar{c}_m\}$ with $\bar{c}_i \in \pi_2$ $(1 \leqq i \leqq m)$. Then $P(a, b, c_1, \ldots, c_m) = 0$ for an irreducible polynomial $P(X, Y, Z_1, \ldots, Z_m)$ with coefficients in F. The polynomial P contains all variables explicitely since a proper subset of a circuit is independent. We write now

$$P(X, Y, Z_1, \ldots, Z_m) = \sum_{i_1, \ldots, i_m} P_{i_1, \ldots, i_m}(X, Y) Z_1^{i_1} \ldots Z_m^{i_m}.$$

The GCD of the coefficients $p_{i_1,\dots,i_m}(X, Y)$ is a constant since P is an irreducible polynomial. By Lemma 1 it follows then that the coefficients have at most a finite number of common zeroes (α, β). We choose $c \in F$ distinct from the second components of these zeroes and such that $P(X, c, c_1, \dots, c_m) \neq 0$ and $Q(X, c) \neq 0$, which is possible because F is infinite. We now have $P(\alpha(c), c, Z_1, \dots, Z_m) \neq 0$ and $P(\alpha(c), c, c_1, \dots, c_m) = 0$. Then if $\alpha(c) \in F$ we find that $\{c_1, \dots, c_m\}$ is algebraically dependent over F, which is impossible since $\{\bar{c}_1, \dots, \bar{c}_m\}$ is a proper subset of a circuit. Therefore $\alpha(c)$ is transcendental over F. Let $P(X, Y) = P(X, Y, c_1, \dots, c_m)$.

We have found $c \in F$ such that $\alpha(c)$ is transcendental over F and $P(\alpha(c), c) = 0$, $Q(\alpha(c), c) = 0$. Since $P(X, c) \neq 0$, $Q(X, c) \neq 0$ and these polynomials have coefficients in the fields π_2 and π_3 respectively, we conclude that $\overline{\alpha(c)} \in \pi_2 \wedge \pi_3 = \pi_4$. We have thus found a point $\bar{x} \in \pi_1 \wedge \pi_2 \wedge \pi_3$. This means that the fields π_1, π_2 and π_3 are extensions of $\overline{F(x)}$. If we replace F by $\overline{F(x)}$ then the ranks of the flats π_i, $\pi_i \vee \pi_j$ etc. are decreased by 1 (we make a contraction by $\overline{F(x)}$). By induction it follows that $\pi_1 \vee \pi_2 \wedge \pi_3$ has transcendence degree $r-4$ over $F(x)$, hence transcendence degree $r-3$ over F, which was to be proved. ∎

A class of non-algebraic matroids

Generalizing the construction of the Vámos matroid, we obtain an infinite class of non-algebraic matroids as follows.

Let $r \geqq 4$. Choose disjoint point-sets S_1, S_2, S_3, S_4 of size $r-2$ and let $S = S_1 \cup S_2 \cup S_3 \cup S_4$. S will be the elements of a paving matroid (cf. [6], p. 40). Let $\mathcal{H}$ be the collection of sets containing $S_1 \cup S_2$, $S_1 \cup S_3$, $S_2 \cup S_3$, $S_2 \cup S_4$, $S_3 \cup S_4$ and all subsets of S size $r-1$ which are not subsets of the first mentioned 5 sets. Then $\mathcal{H}$ contains all hyperplanes of a matroid $P(\mathcal{H})$. The rank of this matroid is r and we find easily $r(S_i) = r-2$, $r(S_i \cup S_j) = r-1$ for $\{i, j\} \neq \{1, 4\}$, $r(S_1 \cup S_4) = r$ and $r(S_1 \cup S_2 \cup S_3) = r(S_2 \cup S_3 \cup S_4) = r$.

We claim that $P(\mathcal{H})$ is non-algebraic. For assume that it is algebraic over a field F. Then we may embed $P(\mathcal{H})$ in an ACCG over F. Let π_i be the closure of S_i in this ACCG for $1 \leqq i \leqq 4$. If we apply the theorem to π_1, π_2, π_3 we find that $\pi_2 \wedge \pi_3 \leqq \pi_1$ and $r(\pi_2 \wedge \pi_3) = r-3$. Using π_2, π_3, π_4 we find similarly $\pi_2 \wedge \pi_3 \leqq \pi_4$. Therefore $\pi_2 \wedge \pi_3 \leqq \pi_1 \wedge \pi_4$ and $r(\pi_1 \wedge \pi_4) \geqq r-3$. Using $r(\pi_1 \vee \pi_4) = r(S_1 \cup S_4) = r$ and $r(\pi_1) = r(\pi_4) = r-2$, we find then $r(\pi_1 \wedge \pi_4) + r(\pi_1 \vee \pi_4) > r(\pi_1) + r(\pi_4)$, which is a contradiction to the submodular inequality. Therefore, the matroid $P(\mathcal{H})$ must be non-algebraic.

We obtain a non-algebraic matroid M_r of rank r for each $r \geqq 4$. The minimum size of a circuit of M_r is r. It follows that M_s can not be a pure restriction minor of M_r when $s < r$. If M_s is still a minor it has to be a minor of a contraction of M_r. Then M_s would be algebraic for we can prove that contractions of M_r are representable as vector matroids and therefore algebraic (minors of algebraic matroids are algebraic). The contradiction implies that M_s can not be a minor of M_r when $s \neq r$.

The contractions M_r/e with $e \in S$ are paving matroids of rank $r-1$. We call a hyperplane dependent when its size is larger than the rank. A flat of rank $r(M)-2$ in a matroid M is called a coline. The paving matroids M_r/e have a particularly simple structure: the dependent hyperplanes contain a fixed coline. It is not hard to see that paving matroids of this type have vector representation over any infinite field. First choose independent vectors representing the coline. Then if $H_1, \ldots, H_m$ are the dependent hyperplanes find representation for the points of H_1, then $H_2, \ldots, H_m$ inductively. Each time we want to add a new point to those which have got a vector representation we have to avoid a finite number of subspaces of codimension 1 spanned by independent hyperplanes among the points which have a representation. When all points of $H_1 \cup \ldots \cup H_m$ have got a representation proceed with those points, which do not belong to a dependent hyperplane. By [6, Theorem 11.2.1] we obtain then an algebraic representation of the matroid. We have thus proved that no matroid M_r contains another one as a minor.

Note that each M_r has to contain a minimal non-algebraic restriction minor M_r'. Any minor of M_r' is algebraic since contractions of M_r are algebraic. It follows that algebraic representability for matroids can not be characterized by a finite number of excluded minors. It would be interesting to know whether the dual matroids M_r^*, $r \geqq 4$, are algebraic. I guess they are non-algebraic. This will follow if the following conjecture is true (by considering the flats spanned by S_1, S_2, S_3 and S_4 as before).

Conjecture. Let $n \geqq 1$ be an integer. Assume that π_1, π_2, π_3 are flats of an ACCG over F such that for some $r \geqq 3n$ we have $r(\pi_1 \vee \pi_2 \vee \pi_3) = r$, $r(\pi_i \vee \pi_j) = r - n$ $(1 \leqq i < j \leqq 3)$, $r(\pi_i) = r - 2n$ $(1 \leqq i \leqq 3)$. Then it follows $\pi_1 \wedge \pi_2 = \pi_1 \wedge \pi_3 = \pi_2 \wedge \pi_3$ and the rank of this flat is $r - 3n$. ▮

In order to make this conjecture plausible we shall prove $\pi_1 \wedge \pi_2 = \pi_1 \wedge \pi_3 = \pi_2 \wedge \pi_3$ using the assumptions and Lemma 3. By the submodular inequality it follows when $\{i, j, k\} = \{1, 2, 3\}$ $r((\pi_i \vee \pi_j) \wedge (\pi_i \vee \pi_k)) \leqq 2(r-n) - r = r - 2n$. Since $r(\pi_i) = r - 2n$ and $\pi_i \leqq (\pi_i \vee \pi_j) \wedge (\pi_i \vee \pi_k)$, we conclude that $(\pi_i \vee \pi_j) \wedge (\pi_i \vee \pi_k) = \pi_i$. Then apply Lemma 3!

Note added in proof. Dress and Lovász have proved the conjecture!

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Bernt Lindström

Dept. of Mathematics,
University of Stockholm
Box 6701, S-113 85 Stockholm, Sweden